

## SUMMABILITY OF MULTILINEAR FORMS ON CLASSICAL SEQUENCE SPACES

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ABSTRACT. We present an extension of the Hardy–Littlewood inequality for multilinear forms. More precisely, let  $\mathbb{K}$  be the real or complex scalar field and  $m, k$  be positive integers with  $m \geq k$  and  $n_1, \dots, n_k$  be positive integers such that  $n_1 + \dots + n_k = m$ .

(a) If  $(r, p) \in (0, \infty) \times [2m, \infty]$  then there is a constant  $D_{m,r,p,k}^{\mathbb{K}} \geq 1$  (not depending on  $n$ ) such that

$$\left( \sum_{i_1, \dots, i_k=1}^n \left| T(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k}) \right|^r \right)^{\frac{1}{r}} \leq D_{m,r,p,k}^{\mathbb{K}} \cdot n^{\max\left\{\frac{2kp-kpr-pr+2rm}{2pr}, 0\right\}} \|T\|$$

for all  $m$ -linear forms  $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$  and all positive integers  $n$ . Moreover, the exponent  $\max\left\{\frac{2kp-kpr-pr+2rm}{2pr}, 0\right\}$  is optimal.

(b) If  $(r, p) \in (0, \infty) \times (m, 2m]$  then there is a constant  $D_{m,r,p,k}^{\mathbb{K}} \geq 1$  (not depending on  $n$ ) such that

$$\left( \sum_{i_1, \dots, i_k=1}^n \left| T(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k}) \right|^r \right)^{\frac{1}{r}} \leq D_{m,r,p,k}^{\mathbb{K}} \cdot n^{\max\left\{\frac{p-rp+rm}{pr}, 0\right\}} \|T\|$$

for all  $m$ -linear forms  $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$  and all positive integers  $n$ . Moreover, the exponent  $\max\left\{\frac{p-rp+rm}{pr}, 0\right\}$  is optimal.

The case  $k = m$  recovers a recent result due to G. Araujo and D. Pellegrino.

## 1. INTRODUCTION

Let  $\mathbb{K}$  be  $\mathbb{R}$  or  $\mathbb{C}$ ,  $m$  be a positive integer and  $p_1, \dots, p_m \in [1, \infty]$ . For  $\mathbf{p} := (p_1, \dots, p_m) \in [1, \infty]^m$ , let

$$\left| \frac{1}{\mathbf{p}} \right| := \frac{1}{p_1} + \dots + \frac{1}{p_m},$$

and let us denote  $X_p := \ell_p$ , when  $1 \leq p < \infty$ , and  $X_\infty := c_0$ . The following problem has been investigated since the 30's and has important applications:

What is the best value of  $\rho$  such that there is a constant  $C_{\rho, \mathbf{p}}^{\mathbb{K}}$  such that

$$(1.1) \quad \left( \sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^\rho \right)^{\frac{1}{\rho}} \leq C_{\rho, \mathbf{p}}^{\mathbb{K}} \|T\|$$

for all continuous  $m$ -linear forms  $T : X_{p_1} \times \dots \times X_{p_m} \rightarrow \mathbb{K}$  and all positive integers  $n$ ?

The answer is divided in some cases; for instance:

- (1)  $\rho = \frac{2m}{m+1}$ , when  $\mathbf{p} = (\infty, \dots, \infty)$  (Bohnenblust–Hille, [6]);
- (2)  $\rho = \frac{2m}{m+1-2\left|\frac{1}{\mathbf{p}}\right|}$ , when  $\left|\frac{1}{\mathbf{p}}\right| \leq \frac{1}{2}$  (Hardy–Littlewood [10] and Praciano-Pereira [13]);
- (3)  $\rho = \frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}$ , when  $\frac{1}{2} \leq \left|\frac{1}{\mathbf{p}}\right| < 1$  (Hardy–Littlewood [10] and Dimant–Sevilla-Peris [8]);
- (4)  $\rho = 1$ , when  $\mathbf{p} = (\infty, \dots, \infty)$  and  $j_1 = \dots = j_m = j$  (Aron and Globevnik [5]);
- (5)  $\rho = \frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|}$ , when  $\left|\frac{1}{\mathbf{p}}\right| < 1$  and  $j_1 = \dots = j_m = j$  (Zalduendo [14]).

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These results were successfully unified in a unique inequality in [2], thanks to consider repeated indexes  $j_k$  in the summands. Let  $n_1, \dots, n_k$  be positive integers and  $n_1 + \dots + n_k = m$ , and let us denote by  $(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k})$  the  $m$ -tuple

$$(e_{i_1}, \overset{n_1 \text{ times}}{\dots}, e_{i_1}, \dots, e_{i_k}, \overset{n_k \text{ times}}{\dots}, e_{i_k}).$$

In [2] the following result is proved:

**Theorem 1** (Albuquerque et al. [2]). *Let  $m \geq k \geq 1$ ,  $m < p \leq \infty$  and let  $n_1, \dots, n_k \geq 1$  be such that  $n_1 + \dots + n_k = m$ . Then, for every continuous  $m$ -linear form  $T : X_p \times \dots \times X_p \rightarrow \mathbb{K}$ , there is a constant  $M(k, m, p, \mathbb{K}) \geq 1$  such that*

$$(1.2) \quad \left( \sum_{i_1, \dots, i_k=1}^{\infty} |T(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k})|^\rho \right)^{\frac{1}{\rho}} \leq M(k, m, p, \mathbb{K}) \|T\|,$$

with

$$(1.3) \quad \rho = \frac{p}{p-m} \text{ for } m < p \leq 2m \text{ and } M(k, m, p, \mathbb{K}) \leq C_{k,p}^{\mathbb{K}}$$

and

$$(1.4) \quad \rho = \frac{2kp}{kp+p-2m} \text{ for } p \geq 2m \text{ and } M(k, m, p, \mathbb{K}) \leq D_{k,p}^{\mathbb{K}}.$$

Moreover, in both cases, the exponent  $\rho$  is optimal.

The optimality of the exponent in (1.2), implies that no constant independent of  $n$  can be found for all  $m$ -linear forms when a smaller exponent  $r$  is considered. Our objective is to show that, even for smaller exponents  $r$ , the value of the left hand sum increases in  $n$  under control, with an explicit dependence on a power factor of  $n$ . We give exactly the optimal exponent for  $n$ . Some previous incursions to this new approach have been done in [7, Corollary 5.20]. However, it is in [3] where this subject has been first explored in its own; there, Hardy-Littlewood type inequalities have been considered and that paper has been the trigger of our work. More recently, in [9] inequalities involving homogeneous polynomials are studied and the asymptotic behavior of the constants whenever the number of variables tends to infinity is established.

This paper is a natural continuation of [2] and, in some sense it is also related to the notion of index of summability introduced by Maia, Pellegrino and Santos [11], which essentially investigates what dependence on  $n$  emerges when we perturb some well known inequalities (for instance in (1.1)).

The main result of the present paper is the following:

**Theorem 2.** *Let  $m, k$  be positive integers,  $m \geq k$ , and let  $n_1, \dots, n_k$  be positive integers such that  $n_1 + \dots + n_k = m$ .*

(a) *If  $(r, p) \in (0, \infty) \times [2m, \infty]$  then there is a constant  $D_{m,r,p,k}^{\mathbb{K}} \geq 1$  (not depending on  $n$ ) such that*

$$\left( \sum_{i_1, \dots, i_k=1}^n |T(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k})|^r \right)^{\frac{1}{r}} \leq D_{m,r,p,k}^{\mathbb{K}} \cdot n^{\max\{\frac{2kp-kpr-pr+2rm}{2pr}, 0\}} \|T\|$$

*for all  $m$ -linear forms  $T : X_p \times \dots \times X_p \rightarrow \mathbb{K}$  and all positive integers  $n$ . Moreover, the exponent  $\max\{\frac{2kp-kpr-pr+2rm}{2pr}, 0\}$  is optimal.*

(b) *If  $(r, p) \in (0, \infty) \times (m, 2m]$  then there is a constant  $D_{m,r,p,k}^{\mathbb{K}} \geq 1$  (not depending on  $n$ ) such that*

$$\left( \sum_{i_1, \dots, i_k=1}^n |T(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k})|^r \right)^{\frac{1}{r}} \leq D_{m,r,p,k}^{\mathbb{K}} \cdot n^{\max\{\frac{p-rp+rm}{pr}, 0\}} \|T\|$$

*for all  $m$ -linear forms  $T : X_p \times \dots \times X_p \rightarrow \mathbb{K}$  and all positive integers  $n$ . Moreover, the exponent  $\max\{\frac{p-rp+rm}{pr}, 0\}$  is optimal.*

## 2. THE PROOF

Let  $E_1, \dots, E_m$  be Banach spaces. The product  $\hat{\otimes}_{j \in \{1, \dots, m\}}^\pi E_j = E_1 \hat{\otimes}^\pi \dots \hat{\otimes}^\pi E_m$  denotes the  $m$ -fold completed projective tensor product of  $E_1, \dots, E_m$ . The tensor  $x_1 \otimes \dots \otimes x_m$  will be denoted by  $\otimes_{j \in \{1, \dots, m\}} x_j$ , and  $\otimes_m x$  shall denote the tensor  $x \otimes \dots \otimes x$ .

Define  $\frac{1}{r_j} = \frac{n_j}{p}$  and note that  $\frac{1}{r_j} < 1$  for all  $j = 1, \dots, k$  (because  $p > m$ ). Let  $D_{r_j} \subset X_p \hat{\otimes}^\pi \dots \hat{\otimes}^\pi X_p$  ( $n_j$  times) be the vector space generated by the tensors  $\otimes_{n_j} e_i$  and consider the isometric isomorphism (see [4] and [2])  $u_j : X_{r_j} \rightarrow \overline{D_{r_j}}$  defined by

$$u_j \left( \sum_{i=1}^{\infty} a_i e_i \right) = \sum_{i=1}^{\infty} a_i \otimes_{n_j} e_i.$$

For any continuous  $m$ -linear form  $T : X_p \times \dots \times X_p \rightarrow \mathbb{K}$ , consider its  $k$ -linearization  $\hat{T} : \otimes_{n_1}^\pi X_p \times \dots \times \otimes_{n_k}^\pi X_p$ , that is,  $\hat{T}$  is the unique  $k$ -linear form such that  $\hat{T}(x_1^1 \otimes \dots \otimes x_{n_1}^1, \dots, x_1^k \otimes \dots \otimes x_{n_k}^k) = T(x_1^1, \dots, x_{n_1}^1, \dots, x_1^k, \dots, x_{n_k}^k)$  for all  $x_j^i \in X_p$ ,  $1 \leq j \leq n_i$ ,  $1 \leq i \leq k$  (for further details we refer to [2]), and let  $S : X_{r_1} \times \dots \times X_{r_k} \rightarrow \mathbb{K}$  be given by

$$S(w_1, \dots, w_k) := \hat{T}(u_1(w_1), \dots, u_k(w_k)).$$

Then

$$(2.1) \quad \left( \sum_{i_1, \dots, i_k=1}^n |T(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k})|^r \right)^{\frac{1}{r}} = \left( \sum_{i_1, \dots, i_k=1}^n |\hat{T}(u_1(e_{i_1}), \dots, u_k(e_{i_k}))|^r \right)^{\frac{1}{r}} \\ = \left( \sum_{i_1, \dots, i_k=1}^n |S(e_{i_1}, \dots, e_{i_k})|^r \right)^{\frac{1}{r}}.$$

**Proof of (a).** Let us first suppose that  $(r, p) \in \left(0, \frac{2kp}{kp+p-2m}\right] \times [2m, \infty]$ . Using the Hölder inequality and Theorem 1 we have

$$\begin{aligned} & \left( \sum_{i_1, \dots, i_k=1}^n |T(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k})|^r \right)^{\frac{1}{r}} \\ & \leq \left( \sum_{i_1, \dots, i_k=1}^n |T(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k})|^{\frac{2kp}{kp+p-2m}} \right)^{\frac{kp+p-2m}{2kp}} \cdot \left( \sum_{i_1, \dots, i_k=1}^n |1|^{\frac{2kpr}{2kp-rkp-rp+2mr}} \right)^{\frac{2kp-rkp-rp+2mr}{2kpr}} \\ & = \left( \sum_{i_1, \dots, i_k=1}^n |T(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k})|^{\frac{2kp}{kp+p-2m}} \right)^{\frac{kp+p-2m}{2kp}} \cdot (n^k)^{\frac{2kp-rkp-rp+2mr}{2kpr}} \\ & \leq M(k, m, p, \mathbb{K}) \|T\| \cdot (n^k)^{\frac{2kp-rkp-rp+2mr}{2kpr}} \\ & = M(k, m, p, \mathbb{K}) \|T\| \cdot n^{\frac{2kp-rkp-rp+2mr}{2pr}}. \end{aligned}$$

On the other hand, if  $(r, p) \in \left[\frac{2kp}{kp+p-2m}, \infty\right] \times [2m, \infty]$ , we have

$$\begin{aligned} & \left( \sum_{i_1, \dots, i_k=1}^n |T(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k})|^r \right)^{\frac{1}{r}} \\ & \leq \left( \sum_{i_1, \dots, i_k=1}^{\infty} |T(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k})|^{\frac{2kp}{kp+p-2m}} \right)^{\frac{kp+p-2m}{2kp}} \\ & \leq M(k, m, p, \mathbb{K}) \|T\| \\ & = M(k, m, p, \mathbb{K}) \|T\| \cdot n^{\max\{\frac{2kp-rkp-rp+2mr}{2pr}, 0\}} \end{aligned}$$

and, of course, in this case the exponent  $\max\left\{\frac{2kp-rkp-rp+2mr}{2pr}, 0\right\}$  is optimal.

It remains to prove the optimality of the exponent in the case  $(r, p) \in \left(0, \frac{2kp}{kp+p-2m}\right] \times [2m, \infty]$ . We shall use a technique used in the main result of [2]. Suppose that  $\lambda \geq 0$  is the smallest exponent satisfying

$$(2.2) \quad \left( \sum_{i_1, \dots, i_k=1}^n |T(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k})|^r \right)^{\frac{1}{r}} \leq D_{m,r,p,k}^{\mathbb{K}} \cdot n^\lambda \|T\|$$

for all continuous  $m$ -linear forms  $T : X_p \times \dots \times X_p \rightarrow \mathbb{K}$ . Let us show that  $\lambda = \max\left\{\frac{2mr+2kp-kpr-pr}{2pr}, 0\right\}$ .

Let  $A : X_{r_1} \times \dots \times X_{r_k} \rightarrow \mathbb{K}$  be a continuous  $k$ -linear form. For each  $i = 1, \dots, k$  we know that  $\overline{D}_{r_i}$  is complemented into  $\hat{\otimes}_{j \in \{1, \dots, m\}}^{\pi} X_p$ , and consider the canonical projection  $d_{r_i} : \hat{\otimes}_{j \in \{1, \dots, m\}}^{\pi} X_p \rightarrow \overline{D}_{r_i}$  (see [4] for details). Defining the  $m$ -linear form  $T_A : X_p \times \dots \times X_p \rightarrow \mathbb{K}$  by

$$T_A(x_1^{(1)}, \dots, x_{n_1}^{(1)}, \dots, x_1^{(k)}, \dots, x_{n_k}^{(k)}) := A(u_{r_1}^{-1} \circ d_{r_1}(x_1^{(1)} \otimes \dots \otimes x_{n_1}^{(1)}), \dots, u_{r_k}^{-1} \circ d_{r_k}(x_1^{(k)} \otimes \dots \otimes x_{n_k}^{(k)})),$$

we have

$$(2.3) \quad \begin{aligned} T_A(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k}) &= A(u_{r_1}^{-1} \circ d_{r_1}(\otimes_{n_1} e_{i_1}), \dots, u_{r_k}^{-1} \circ d_{r_k}(\otimes_{n_k} e_{i_k})) \\ &= A(u_{r_1}^{-1}(\otimes_{n_1} e_{i_1}), \dots, u_{r_k}^{-1}(\otimes_{n_k} e_{i_k})) = A(e_{i_1}, \dots, e_{i_k}). \end{aligned}$$

By (2.3) and (2.2) applied to  $T_A$ , and using that  $\|T_A\| \leq \|A\|$ , we obtain

$$\left( \sum_{i_1, \dots, i_k=1}^n |A(e_{i_1}, \dots, e_{i_k})|^r \right)^{\frac{1}{r}} \leq D_{m,r,p,k}^{\mathbb{K}} \cdot n^\lambda \|A\|.$$

Since  $A$  is  $k$ -linear, and

$$\frac{1}{r_j} = \frac{n_j}{p} \leq \frac{m}{2m} = \frac{1}{2},$$

from the Kahane–Salem–Zygmund inequality (see [1, Lemma 6.1] for details), there is a constant  $C_k > 0$  such that

$$n^{\frac{k}{r}} \leq C_k M_{k,r_1, \dots, r_k}^{\mathbb{K}} n^\lambda n^{\frac{k+1}{2} - (\frac{1}{r_1} + \dots + \frac{1}{r_k})}.$$

Making  $n \rightarrow \infty$ , we have

$$\lambda \geq \frac{1}{r_1} + \dots + \frac{1}{r_k} + \frac{2k - kr - r}{2r}.$$

Since  $\frac{1}{r_1} + \dots + \frac{1}{r_k} = \frac{m}{p}$ , we have

$$\lambda \geq \frac{m}{p} + \frac{2k - kr - r}{2r} = \max\left\{\frac{2mr + 2kp - kpr - pr}{2pr}, 0\right\}.$$

**Proof of (b).** Since

$$\left( \sum_{i_1, \dots, i_k=1}^n |T(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k})|^r \right)^{\frac{1}{r}} \leq \left( \sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^r \right)^{\frac{1}{r}},$$

by [3, Theorem 1.1(b)] we have

$$\left( \sum_{i_1, \dots, i_k=1}^n |T(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k})|^r \right)^{\frac{1}{r}} \leq D_{m,r,p,k}^{\mathbb{K}} \cdot n^{\max\left\{\frac{p-rp+rm}{pr}, 0\right\}} \|T\|.$$

Let us prove the optimality of the exponent. If

$$\frac{mr + p - pr}{pr} \leq 0$$

the optimality of the exponent  $\max\{(mr + p - pr)/pr, 0\}$  is immediate.

Suppose that the inequality holds for a certain exponent  $s \geq 0$ ; thus

$$(2.4) \quad \left( \sum_{i_1, \dots, i_k=1}^n |T(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k})|^r \right)^{\frac{1}{r}} \leq D_{m,r,p,k}^{\mathbb{K}} \cdot n^s \|T\|.$$

As in the previous case, for each continuous  $k$ -linear form  $A : X_{r_1} \times \cdots \times X_{r_k} \rightarrow \mathbb{K}$ , with  $r_j = \frac{p}{n_j}$ , and for all  $j = 1, \dots, k$ , there is a continuous  $m$ -linear form  $T_A : X_p \times \cdots \times X_p \rightarrow \mathbb{K}$  such that

$$(2.5) \quad T_A(e_{i_1}^{n_1}, \dots, e_{i_k}^{n_k}) = A(e_{i_1}, \dots, e_{i_k})$$

and  $\|T_A\| \leq \|A\|$ . By (2.5) and (2.4) applied to  $T_A$  we obtain

$$(2.6) \quad \left( \sum_{i_1, \dots, i_k=1}^n |A(e_{i_1}, \dots, e_{i_k})|^r \right)^{\frac{1}{r}} \leq D_{m,r,p,k}^{\mathbb{K}} \cdot n^s \|T_A\| \leq D_{m,r,p,k}^{\mathbb{K}} \cdot n^s \|A\|.$$

Define the  $k$ -linear form  $S : X_{r_1} \times \cdots \times X_{r_k} \rightarrow \mathbb{K}$  by

$$S(x^{(1)}, \dots, x^{(k)}) = \sum_{j=1}^n x_j^{(1)} \cdots x_j^{(k)},$$

and notice that by the Hölder inequality we have

$$\|S\| \leq n^{1 - \left(\frac{1}{r_1} + \cdots + \frac{1}{r_k}\right)} = n^{1 - \frac{m}{p}}.$$

Therefore, plugging  $S$  into (2.6) we get

$$n^{\frac{1}{r}} \leq D_{m,r,p,k}^{\mathbb{K}} n^s n^{1 - \frac{m}{p}}$$

and we easily conclude that

$$s \geq \frac{p - rp + rm}{pr}.$$

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